Modeling and Optimization Methods for Reliability Experiments

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Abstract

We propose a new classification of noise factors for the use in experiments with degradation data. Based on the above classification and using a stochastic differential equation for the degradation rate we obtain a statistical model for the degradation measurement. We then develop a two-step optimization procedure for finding the optimal control factor setting. The methodology is illustrated using a fluorescent lamp experiment.

1 Introduction

Designed experiments are widely used in industries for quality improvement. Surprisingly, not much work have been done on applying the above important statistical tool for reliability improvement. Several case studies are reported in the literature, for example, see Tseng, Hamada, and Chiao (1995) and Condra (2001). They clearly demonstrate the importance of design of experiments for reliability improvement. A recent review on this topic is given by Nair, Escobar, and Hamada (2003).

Two types of data are usually encountered in reliability experiments: life time data and degradation data. In this short paper we describe our work on experiments with degradation data.

2 Degradation data

Degradation measurement is a commonly used data type in reliability analysis. It is a measurement which is highly related to the lifetime, like the luminosity to a fluorescent light bulb (Tseng, Hamada, and Chiao 1995), the fatigue crack-size in a metal (Hudak et al., 1978). Usually, the failure of a product can be traced to the degradation process. The degradation measurements are always monotonic and when it crosses a certain value (the threshold) the product fails. If the degradation measurement is available, it provides more information than the life time data.

2.1 Classification of factors

In robust parameter design, factors are classified as control and noise factors. Control factors can be easily controlled but noise factors are either difficult or impossible to control during the normal user conditions. Different types of noise factors are discussed in Wu and Hamada (2000, chapter 10). For the use in reliability experiments we classify the noise factors into two: product noise and environmental noise. *Product noise* factors are those factors that vary from product to product. For example, the resistance of the filament in a light bulb will be different from unit to unit. *Environmental noise* factors are those factors that vary during the usage of the product. For example, temperature and humidity around the light bulb can vary during its usage. We make this classification because, as we will see, the variation introduced by them on the degradation measurements have different structures. In this article, we use $\boldsymbol{X}$ to denote the control factors, $\boldsymbol{N}$ to the product noises and $\boldsymbol{Q}_t$ to the environmental noises. Note that we index the environmental noises using the time t because they vary over time.

2.2 Modeling

Let Y_t be the degradation measurement at time t. Assume that the degradation rate, may be after a suitable transformation of the original degradation measurement, is constant, i.e. $dY_t/dt = constant$. But this constant depends on the control and noise factors. We assume the following model:

$$\frac{dY_t}{dt} = \beta(\boldsymbol{X}, \boldsymbol{N}) + W_t(\boldsymbol{X}, \boldsymbol{N}, \boldsymbol{Q}_t),$$

where $\beta(\boldsymbol{X}, \boldsymbol{N})$ is the rate of degradation and $W_t(\boldsymbol{X}, \boldsymbol{N}, \boldsymbol{Q}_t)$ is the zero mean noise term. Since the degradation path is a continuous path we cannot assume the noise term to be independent over time. Assume that $W_t(\boldsymbol{X}, \boldsymbol{N}, \boldsymbol{Q}_t)dt = \sigma(\boldsymbol{X}, \boldsymbol{N}, \boldsymbol{Q}_t)dB_t$, where B_t is a standard Brownian motion. This will give continuous sample paths for Y_t. If $Y_0 = 0$, then

$$Y_t = \int_0^t \beta(\boldsymbol{X}, \boldsymbol{N})ds + \int_0^t \sigma(\boldsymbol{X}, \boldsymbol{N}, \boldsymbol{Q}_s)dB_s$$

It can be proved by using some properties of Ito integrals (see, for example, Øksendal, 2003 chapter 3) that

$$E[Y_t|\boldsymbol{N}] = \beta(\boldsymbol{X}, \boldsymbol{N})t$$

and

$$\begin{aligned} Var[Y_t|\boldsymbol{N}] &= [\textstyle\int_0^t \sigma(\boldsymbol{X}, \boldsymbol{N}, \boldsymbol{Q}_s)dB_s|\boldsymbol{N}] \\ &= E[(\textstyle\int_0^t \sigma(\boldsymbol{X}, \boldsymbol{N}, \boldsymbol{Q}_s)dB_s)^2|\boldsymbol{N}] \\ &= \textstyle\int_0^t E(\sigma(\boldsymbol{X}, \boldsymbol{N}, \boldsymbol{Q}_s)^2|\boldsymbol{N})ds. \end{aligned}$$

Furthermore, assume that $E(\sigma(\boldsymbol{X}, \boldsymbol{N}, \boldsymbol{Q}_t)^2|\boldsymbol{N}) = \sigma^2(\boldsymbol{X}, \boldsymbol{N})$, then the expectation and variance of the degradation measurement at time t are

$$\begin{aligned} E[Y_t] &= EE[Y_t|\boldsymbol{N}] \\ &= E[\beta(\boldsymbol{X}, \boldsymbol{N})]t, \end{aligned}$$

and

$$\begin{aligned} Var[Y_t] &= EVar[Y_t|\boldsymbol{N}] + VarE[Y_t|\boldsymbol{N}] \\ &= E[\sigma^2(\boldsymbol{X}, \boldsymbol{N})]t + Var[\beta(\boldsymbol{X}, \boldsymbol{N})]t^2 \end{aligned}$$

respectively.

2.3 Optimization and Estimation

Assume, after a suitable transformation, that Y_t is nonnegative and increasing with t. Then Y_t is a smaller-the-better (STB) characteristic. Let τ be the threshold for the failure. Thus the product fails when $y_t > \tau$. It is very common in the reliability literature to assume that the product imparts a loss when it fails and no loss when it functions. But this is not a realistic assumption. The loss should increase over time because the product performance deteriorates with time. We can use the degradation measurement as an indicator of the product's performance. We can define the quality loss at time t as $L(Y_t) = cY_t$, where c is a cost-related coefficient. See Joseph (2004) for more details about this loss function.

Our objective is to find a control factor setting that will minimize the expected loss. Thus we want

$$\min_{\boldsymbol{X}} E\{L(Y_t)\} = E(cY_t) = cE\{\beta(\boldsymbol{X}, \boldsymbol{N})\}t,$$

for all t, which can be achieved by minimizing $E\{\beta(\boldsymbol{X}, \boldsymbol{N})\}$. In other words, we want to find a control factor setting to minimize the average degradation rate. The above procedure does not directly minimize the variation in Y_t. Is it beneficial to do that? By reducing the variation in lifetime without changing the mean, we will be able to increase the lifetime of some products but at the same time the lifetime of some other products will decrease. The short-lived products can seriously damage the reputation of the manufacturer.

So if we prefer increasing the lifetime of short-lived products at the expense of decreasing the lifetime of some long-lived products, then minimizing the variation in lifetime makes sense. It is easy to show that the variation in lifetime can be minimized by minimizing the variation in Y_t.

Let $\boldsymbol{X} = \{\boldsymbol{X}_1, \boldsymbol{X}_2\}$, where $\boldsymbol{X}_1$ is the set of control factors affecting the average degradation rate and $\boldsymbol{X}_2$ the remaining control factors. Since minimizing the average degradation rate is more important than minimizing the variation, first we will find an $\boldsymbol{X}_1^*$ by minimizing $E\{\beta(\boldsymbol{X}_1, \boldsymbol{N})\}$. The optimal setting of the remaining factors will then be obtained by minimizing the variation

$$E[\sigma^2(\boldsymbol{X}_1^*, \boldsymbol{X}_2, \boldsymbol{N})]t + Var[\beta(\boldsymbol{X}_1^*, \boldsymbol{X}_2, \boldsymbol{N})]t^2.$$

This can be done uniformly over t, if $\boldsymbol{X}_2$ do not have conflicting effects on $E[\sigma^2(\boldsymbol{X}_1^*, \boldsymbol{X}_2, \boldsymbol{N})]$ and $Var[\beta(\boldsymbol{X}_1^*, \boldsymbol{X}_2, \boldsymbol{N})]$. But if there is a conflicting effect, we need to select a t to evaluate the variance of Y_t. We will choose this time as the average life time, which is $\bar{t} = \tau / E\{\beta(\boldsymbol{X}_1^*, \boldsymbol{N})\}$. Thus we have the following two-step procedure:

1. Find $\boldsymbol{X}_1^*$ to minimize $E[\beta(\boldsymbol{X}_1, \boldsymbol{N})]$ and let $\bar{t} = \tau / E\{\beta(\boldsymbol{X}_1^*, \boldsymbol{N})\}$.
2. Find $\boldsymbol{X}_2^*$ to minimize $E[\sigma^2(\boldsymbol{X}_1^*, \boldsymbol{X}_2, \boldsymbol{N})]\bar{t} + Var[\beta(\boldsymbol{X}_1^*, \boldsymbol{X}_2, \boldsymbol{N})]\bar{t}^2$.

In practice, $Var[\beta(\boldsymbol{X}, \boldsymbol{N})], E[\beta(\boldsymbol{X}, \boldsymbol{N})]^2$ and $E[\sigma^2(\boldsymbol{X}, \boldsymbol{N})]$ are unknown and need to be estimated. Consider a experiment with I control-factor configurations, $\boldsymbol{X}_1, ...\boldsymbol{X}_I$ and J noise-factor combinations $\boldsymbol{N}_{1(i)}, ...\boldsymbol{N}_{J(i)}$ in each control-factor configuration, say i. For each factor combination, we collect observations at times $(t_1, ..., t_K)$. Then the experimental outcome can be modelled as

$$Y_{ijk} = \beta_{ij} t + \epsilon_{ijk},$$

for $i = 1, ..., I, j = 1, ..., J$, and $k = 1, ..., K$. Under the assumptions that for each $\boldsymbol{X}_i$ and $\boldsymbol{N}_{j(i)}$ the Brownian motion involved are independent from each other and $E(\sigma(\boldsymbol{X}, \boldsymbol{N}, \boldsymbol{Q}_t)^2 | \boldsymbol{N}) = \sigma^2(\boldsymbol{X}, \boldsymbol{N})$, we obtain

$$E[Y_{ijk} | ij] = \beta_{ij} t,$$

and

$$Cov[Y_{ijk_1}, Y_{ijk_2} | ij] = \sigma_{ij}^2 min(t_{k_1}, t_{k_2}),$$

where $\beta_{ij} = \beta(\boldsymbol{X}_i, \boldsymbol{N}_{j(i)})$ and $\sigma_{ij}^2 = \sigma^2(\boldsymbol{X}_i, \boldsymbol{N}_{j(i)})$.

Thus, the parameters can be estimated using generalized least squares and let $\widehat{\beta}_{ij}$ and $\widehat{\sigma}_{ij}^2$ denote the estimators. Then for each control-factor configuration, $E[Y_t]$ and $Var[Y_t]$ can be estimated by

$$\widehat{E[Y_t]} = \overline{\widehat{\beta}_{j.}} t,$$

$$\widehat{Var[Y_t]} = \frac{t}{J} \sum_{j=1}^{J} \widehat{\sigma}_{ij}^2 + \frac{t^2}{J-1} \sum_{j=1}^{J} (\widehat{\beta}_{ij} - \overline{\widehat{\beta}_{j.}})^2,$$

where $\overline{\widehat{\beta}_{j.}} = \frac{1}{J} \sum_{j=1}^{J} \widehat{\beta}_{ij}$.

2.4 An Example

We use the fluorescent lamps experiment in Tseng, Hamada, and Chiao (1995) to illustrate our estimation and optimization procedure. It can also be found in Wu and Hamada (2000, chapter 12). In this experiment, three factors are studied which are A: the amount of electric current in the exhaustive process, B: the concentration of the mercury dispenser in the mercury dispenser coating process, and C: the concentration. They are all control factors. A 2^{3-1} fractional factorial design with the defining relation $\boldsymbol{I} = ABC$, is applied and there are 5 replicates in each factor configuration, which are run 1:(-,-,-), run 2:(-,+,+), run 3:(+,-,+), and run 4:(+,+,-).

The estimates of β_{ij} and σ^2_{ij} are obtained as described in Section 2.3 and are given in the table below. Then regressing with respect to the control factors, we obtain

$$\widehat{E[\beta]} = 4.2442 - 1.2695B - 0.5396C,$$

$$\widehat{E[\sigma^2]} = 1.7527 - 1.2085C,$$

$$\widehat{Var[\beta]} = 1.5904 - 1.0613C.$$

Setting both factors B and C to "high" minimizes all the above three quantities. Since there is no conflict for the optimal setting a two-step optimization procedure is not required in the above example.

Run	$\widehat{\beta_{ij}}(10^{-6})$	$\widehat{\sigma^2_{ij}}(10^{-6})$	$\widehat{\widehat{\beta_{i.}}}(10^{-6})$	$S^2_{\widehat{\beta_{i.}}}(10^{-13})$	$\widehat{\widehat{\sigma^2_{i.}}}(10^{-6})$	$coef.(t)(10^{-6})$	$coef.(t^2)(10^{-11})$
	3.87	2.46					
	3.79	2.61					
1	4.27	1.34	4.24	1.64	2.12	2.12	1.82
	4.68	1.54					
	4.60	2.62					
	2.15	0.32					
	2.44	0.62					
2	2.44	0.78	2.43	0.29	0.54	0.54	0.60
	2.60	0.54					
	2.53	0.43					
	3.56	0.42					
	3.90	1.02					
3	4.27	0.36	3.71	1.54	0.55	0.55	1.39
	3.56	0.35					
	3.24	0.58					
	2.79	1.53					
	3.39	1.48					
4	3.12	1.10	2.98	0.77	1.39	1.39	0.89
	2.72	1.71					
	2.86	1.32					

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